CHAPTER 16

GRADIENTS, MAXIMA, AND MINIMA

16.1 Gradients and Directional Derivatives

PREREQUISITES

1. Recall how to use the chain rule for several variables (Section 15.3).

2. Recall how to compute a dot product (Section 13.4).

3. Recall how to draw vectors (Section 13.1).

PREREQUISITE QUIZ

1. Compute the following dot products:

   (a) $(\underline{i} + 2\underline{j} - \underline{k})\cdot(3\underline{i} - \underline{j} + \underline{k})$

   (b) $(\underline{i} - 3\underline{j})\cdot(2\underline{j} + 3\underline{k})$

2. If $\underline{u}$ and $\underline{v}$ are non-zero vectors and $\underline{u}\cdot\underline{v} = 0$, how is $\underline{u}$ geometrically related to $\underline{v}$?

3. Suppose that $f(x,y,z) = xy \sin z$ and $(x,y,z) = (t,t^3,t^2)$. Use the chain rule to find df/dt.

4. If P is the point $(1,2)$, draw the vector $\overrightarrow{PQ} = \underline{i} - \underline{j}$.

GOALS

1. Be able to compute the gradient of a given function.

2. Be able to compute a directional derivative and understand its meaning.

3. Be able to determine the direction of the most rapid or least rapid rate of change.

STUDY HINTS

1. Vector field. This is a function which describes a set of vectors which depend upon their location in the plane or in space. Graphically, we draw these vectors with their tails at the given point in the domain. See Fig. 16.1.1.

2. Gradients. The gradient of a function is a vector field. The gradient operation is denoted by the symbol ∇ . Sometimes, it is denoted grad. Thus, $\nabla f = \text{grad } f$ is the gradient of f . The components of the gradient are simply the partial derivatives of the given function, so $\nabla f = (\partial f/\partial x)\mathbf{i} + (\partial f/\partial y)\mathbf{j} + (\partial f/\partial z)\mathbf{k}$. Memorize the formula for the gradient.

3. Chain rule. The first paragraph of Section 16.1 explains how the chain rule may be thought of as a dot product. It is the gradient of the function dotted with a velocity vector. Thus, $(d/dt)f(\sigma(t)) = \nabla f(\sigma(t))\cdot\sigma'(t)$.

4. Directional derivatives. This concept tells you that a rate of change depends on the direction of movement. For example, if you were climbing a hill, the rate of change of height is greatest going straight up rather than sideways. The computational formula is $\nabla f(\mathbf{r})\cdot\mathbf{d}$, where $\mathbf{d}$ is a unit vector. Memorize this formula. Forgetting that $\mathbf{d}$ is a unit vector is a common mistake. Finally, the directional derivative is a scalar, not a vector.

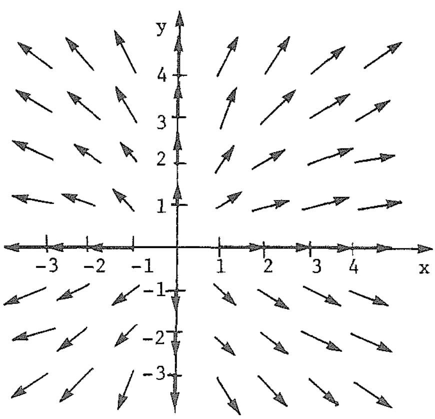
5. Gradient interpretation. Since the directional derivative is $\nabla f(\mathbf{r})\cdot\mathbf{d} = \|\nabla f(\mathbf{r})\|\cdot\|\mathbf{d}\| \cos\theta = \|\nabla f(\mathbf{r})\| \cos\theta$, the maximum rate of change occurs when $\cos\theta$ is maximized. This occurs when $\theta = 0$. Thus, the greatest increase occurs in the direction of ∇f and the greatest decrease occurs in the direction of $-\nabla f$.

6. Partial derivatives. Notice that the directional derivative of f in the direction of $\underline{i}$, $\underline{j}$, or $\underline{k}$ is simply the corresponding partial derivative, f_x, f_y, or f_z.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The gradient of $\underline{\nabla}f(x,y,z) = f_x\underline{i} + f_y\underline{j} + f_z\underline{k}$. $f_x(x,y,z) = x/\sqrt{x^2+y^2+z^2}$, $f_y(x,y,z) = y/\sqrt{x^2+y^2+z^2}$, and $f_z(x,y,z) = z/\sqrt{x^2+y^2+z^2}$. Thus, $\underline{\nabla}f(x,y,z) = (x/\sqrt{x^2+y^2+z^2})\underline{i} + (y/\sqrt{x^2+y^2+z^2})\underline{j} + (z/\sqrt{x^2+y^2+z^2})\underline{k}$.

5. The gradient is $\underline{\nabla}f(x,y) = f_x\underline{i} + f_y\underline{j}$. $\ln\sqrt{x^2+y^2}$ simplifies to $(1/2)\ln(x^2+y^2)$, so $f_x = x/(x^2+y^2)$ and $f_y = y/(x^2+y^2)$. Thus, $\underline{\nabla}f(x,y) = [x/(x^2+y^2)]\underline{i} + [y/(x^2+y^2)]\underline{j}$.

9. 

$f_x = x/4$ and $f_y = y/6$, so the gradient vector field is $(x/4)\underline{i} + (y/6)\underline{j}$. At each point (x_0,y_0), we sketch $\underline{\nabla}f(x_0,y_0)$.

13. $r = \sqrt{x^2+y^2+z^2}$, so $(\partial/\partial x)(1/(x^2+y^2+z^2)) = -2x/(x^2+y^2+z^2)^2$. Similarly, $(\partial/\partial y)(1/(x^2+y^2+z^2)) = -2y/(x^2+y^2+z^2)^2$ and $(\partial/\partial z)(1/(x^2+y^2+z^2)) = -2z/(x^2+y^2+z^2)^2$. Thus, $\underline{\nabla}(1/r^2) = -2(x\underline{i} + y\underline{j} + z\underline{k})/(x^2+y^2+z^2)^2 = -2\underline{r}/r^4$.

17. We want to show that $(d/dt)(f(\underline{\sigma}(t)) = \underline{\nabla}f(\underline{\sigma}(t))\cdot\underline{\sigma}'(t)$. $f_x = (1/2)(2x) \times (x^2 + y^2 + z^2)^{-1/2} = x/\sqrt{x^2 + y^2 + z^2}$. Similarly, we get $\underline{\nabla}f(x,y,z) = (x/\sqrt{x^2 + y^2 + z^2})\underline{i} + (y/\sqrt{x^2 + y^2 + z^2})\underline{j} + (z/\sqrt{x^2 + y^2 + z^2})\underline{k}$, so $\underline{\nabla}f(\underline{\sigma}(t)) = (\sin t/\sqrt{1 + t^2})\underline{i} + (\cos t/\sqrt{1 + t^2})\underline{j} + (t/\sqrt{1 + t^2})\underline{k}$. Also, $\underline{\sigma}'(t) = (\cos t,-\sin t,1)$, and $\underline{\nabla}f(\underline{\sigma}(t))\cdot\underline{\sigma}'(t) = (\sin t \cos t - \cos t \sin t + t)/\sqrt{1 + t^2} = t/\sqrt{1 + t^2}$. On the other hand, $f(\underline{\sigma}(t)) = \sqrt{1 + t^2}$, so the derivative is $(1/2)(1 + t^2)^{-1/2}(2t) = t/\sqrt{1 + t^2}$, which matches the result above.

21. The directional derivative at $\underline{r}$ in the direction $\underline{d}$ is $\underline{\nabla}f(\underline{r})\cdot\underline{d}$, where $\underline{d}$ is a unit vector. We have $f_x(1,2) = (2x - 3y^3)|_{(1,2)} = 2 - 24 = -22$ and $f_y(1,2) = (2y - 9xy^2)|_{(1,2)} = 4 - 36 = -32$. Then the directional derivative is $(-22,-32)\cdot(1/2,\sqrt{3}/2) = -11 - 16\sqrt{3}$.

25. Using the method of Exercise 21, we have $f_x(\underline{r}) = (2x - 2y)|_{(1,1,2)} = 0$, $f_y(\underline{r}) = -2x|_{(1,1,2)} = -2$, and $f_z(\underline{r}) = 6z|_{(1,1,2)} = 12$. Then the directional derivative is $(0,-2,12)\cdot(1,1,-1)/\sqrt{3} = (0 - 2 - 12)/\sqrt{3} = -14/\sqrt{3}$.

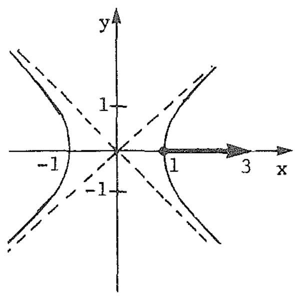
29. The direction of fastest increase occurs in the direction of the gradient. $f_x(1,1) = 2x|_{(1,1)} = 2$; $f_y(1,1) = 4y|_{(1,1)} = 4$. Hence, $2\underline{i} + 4\underline{j}$ is the desired direction. $(\underline{i} + 2\underline{j})/\sqrt{5}$ is the unit vector in the desired direction.

33. (a) Note that $T(1,1,1) = e^{-6}$. Then $T_x(1,1,1) = -2xT|_{(1,1,1)} = -2e^{-6}$; $T_y(1,1,1) = -4yT|_{(1,1,1)} = -4e^{-6}$; and $T_z(1,1,1) = -6zT|_{(1,1,1)} = -6e^{-6}$. The gradient gives the direction of most rapid increase, so she should proceed in the opposite direction, i.e., along $(1,2,3)$.

33. (b) Let $\underline{v} = a(1,2,3)$ denote her velocity. Then $e^8 = \|\underline{v}\| = a\sqrt{14}$, so $a = e^8/\sqrt{14}$. By the chain rule, $dT/dt = T_x(dx/dt) + T_y(dy/dt) + T_z(dz/dt) = (-2e^{-6}e^8/\sqrt{14})(1 + 4 + 9) = -2\sqrt{14}\, e^2$.

(c) We wish to have $dT/dt = -\sqrt{14}e^2$. In terms of the gradient, $dT/dt = \underline{\nabla}T\cdot\underline{\sigma}'(t)$, where $\underline{\sigma}(t)$ is her trajectory. However, $\underline{\nabla}T\cdot\underline{\sigma}'(t) = \|\underline{\nabla}T\|\|\underline{\sigma}'(t)\| \cos\theta$. At $(1,1,1)$, this equals $2e^{-6}(1 + 4 + 9)^{1/2}e^8 \cos\theta = 2\sqrt{14}\, e^2 \cos\theta$, so we need $\cos\theta = 1/2$, where θ is the angle between $-\underline{\nabla}T(1,1,1)$ (along $(1,2,3)$ — see part (a)) and $\sigma'(0)$. Hence $\theta = \pi/6$ and she should fly outside the cone with vertex at $(1,1,1)$, axis along $(1,2,3)$, and sides at an angle of $\pi/6$ from the axis.

37.



$f_x = 2x$ and $f_y = -2y$, so $\underline{\nabla}f(1,0) = 2\underline{i}$, which is the direction of fastest increase.

41. (a) If $f(x,y,z)$ is constant, then $f_x = f_y = f_z = 0$, so $\underline{\nabla}f = \underline{0}$.

(b) Let $h(x,y,z) = f(x,y,z) + g(x,y,z)$. Then $h_x = f_x + g_x$; $h_y = f_y + g_y$; $h_z = f_z + g_z$, so $\underline{\nabla}(f + g) = \underline{\nabla}h = \underline{\nabla}f + \underline{\nabla}g$.

(c) Let $g(x,y,z) = cf(x,y,z)$. Then $g_x = cf_x$; $g_y = cf_y$; $g_z = cf_z$, so $\underline{\nabla}(cf) = \underline{\nabla}g = c\underline{\nabla}f$.

(d) Let $h(x,y,z) - f(x,y,z)\, g(x,y,z)$. Then $h_x = f_xg + fg_x$; $h_y = f_yg + fg_y$; $h_z = f_zg + fg_z$. Therefore $\underline{\nabla}(fg) = \underline{\nabla}h = g\underline{\nabla}f + f\underline{\nabla}g$.

(e) Let $h(x,y,z) = f(x,y,z)/g(x,y,z)$, for $g \neq 0$. Then $h_x = (f_xg - fg_x)/g^2$; $h_y = (f_yg - fg_y)/g^2$; $h_z = (f_zg - fg_z)/g^2$. Therefore, $\underline{\nabla}(f/g) = \underline{\nabla}h = (g\underline{\nabla}f - f\underline{\nabla}g)/g^2$.

45. Let $\underline{\nabla} f(x,y) = (g(x,y), h(x,y))$, $\underline{d}_1$ be the unit vector from the point $(1,3)$ to $(2,3)$, i.e., $\underline{d}_1 = (1,0)$; and $\underline{d}_2$ be the unit vector from the point $(1,3)$ to $(1,4)$, i.e., $\underline{d}_2 = (0,1)$. We know that the directional derivative at $(1,3)$ in the direction $\underline{d}_1$ is $\underline{\nabla} f(1,3)\cdot\underline{d}_1 = 2$, i.e., $g(1,3) = 2$. Also $\underline{\nabla} f(1,3)\cdot\underline{d}_2 = -2$, i.e., $h(1,3) = -2$. Hence, $\underline{\nabla} f(1,3) = (g(1,3),h(1,3)) = (2,-2)$. Thus, the directional derivative in the direction toward $(3,6)$ is $(2,-2)\cdot\underline{d}_3$, where $\underline{d}_3 = [(3,6) - (1,3)]/\|(3,6) - (1,3)\| = (2/\sqrt{13}, 3/\sqrt{13})$. So $(2,-2)\cdot\underline{d}_3 = -2/\sqrt{13}$.

49. $\underline{\nabla} f(x,y) = f_x\underline{i} + f_y\underline{j}$, so $\underline{k}\times\underline{\nabla} f = -f_y\underline{i} + f_x\underline{j}$. If this is a gradient, then the mixed partials should be equal. Differentiate the $\underline{i}$ component of $\underline{k}\times\underline{\nabla} f$ with respect to y to get $-f_{yy}$. Differentiating the $\underline{j}$ component with respect to x gives f_{xx} . Thus, we need $f_{xx} = -f_{yy}$. (The equation $f_{xx} + f_{yy} = 0$ is called Laplace's equation.) This is satisfied by any function of the form $ax + bxy + cy + d$, where a , b , c , and d are constant. This is also satisfied by other special functions such as $f(x,y) = ax^2 + bxy - ay^2 + d$.

SECTION QUIZ

1. Find the direction of fastest decrease for the following:

(a) $f(x,y) = xy + y^3$ at $(1,2)$

(b) $f(x,y) = \cos xy + e^x - x$ at $(-1,0)$

2. The directional derivative for $f(x,y) = x^2 + y^2$ in the direction $\underline{i} + \underline{j}$ at the point $(1,1)$ is not $(2,2)\cdot(1,1) = 4$.

(a) Why is the calculation incorrect?

(b) What is the correct directional derivative?

(c) Explain what the answer in part (b) tells you geometrically.

3. If $u(x,y) = xy^2 + x^2y + 4$ and $(x,y) = (t + 1, \sin t)$, use the gradient to find du/dt.

4. A law-abiding champion downhill skier finds herself on an American snow-covered slope whose shape near $(-1,1)$ approximates that of the surface $f(x,y) = 25 - x^4 - y^2$. She has determined that her speed exceeds the 55 MPH speed limit if the directional derivative at a point is less than -2.

   (a) Does she need to worry about speeding if she skis along the direction $-\underline{i} - \underline{j}$? Explain.

   (b) In what direction is her speed greatest?

   (c) In what directions may she ski and still obey the speed limit?

   (d) Can she ski in the same directions as in part (c) and still be travelling at less than 55 MPH at $(-2,2)$?

ANSWERS TO PREREQUISITE QUIZ

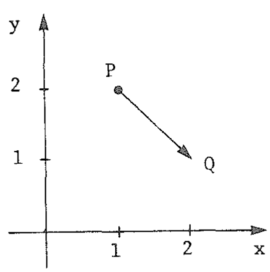
1. (a) 0

   (b) -6

2. They are orthogonal.

3. $4t^3 \sin t^2 + 2t^5 \cos t^2$

4.



ANSWERS TO SECTION QUIZ

1. (a) $-2\underline{i} - 13\underline{j}$

   (b) $(1 - 1/e)\underline{i}$

2. (a) $(1,1)$ should be replaced by the unit vector $(1/\sqrt{2}, 1/\sqrt{2})$.

   (b) $2\sqrt{2}$

   (c) f rises by $2\sqrt{2}$ units when one goes one unit in the direction of $\underline{i} + \underline{j}$.

3. $\sin^2 t + 2(t + 1)(\sin t)(1 + \cos t) + (t + 1)^2 \cos t$

4. (a) No; directional derivative is $-\sqrt{2}$.

   (b) $4\underline{i} - 2\underline{j}$

   (c) Anywhere except the directions from $-\underline{j}$ to $4\underline{i} + 3\underline{j}$, moving counterclockwise.

   (d) No.

16.2 Gradients, Level Surfaces, and Implicit Differentiation

PREREQUISITES

1. Recall how to compute tangent planes to graphs (Section 15.2).
2. Recall how to compute the gradient of a function (Section 16.1).
3. Recall how to differentiate implicitly (Section 2.3).

PREREQUISITE QUIZ

1. Find the equation of the tangent plane to the graph of $z = 2xy + \cos y$ at the point $(1,0,1)$.
2. Find the equation of the tangent plane to the graph of $z = 2xy + \cos y$ at the point $(0,\pi/2,0)$.
3. Compute $\underline{\nabla}f$ if $f(x,y) = 2xy + \cos y$.
4. Compute $\underline{\nabla}f$ if $f(x,y,z) = 3x^2 + yz - z^3$.
5. Find dy/dx if $x^2 + y^2 = 3 + 2y$.

GOALS

1. Be able to find a tangent plane by using gradients.
2. Be able to differentiate a multivariable expression implicitly.
3. Be able to solve related rate problems involving several variables.

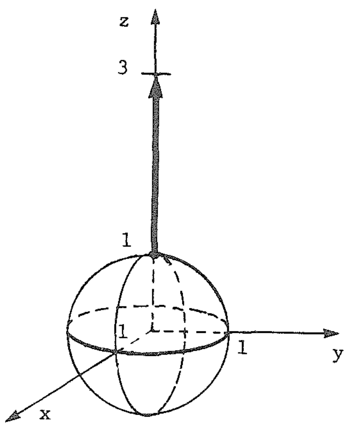
STUDY HINTS

1. Tangent plane to surface. The tangent plane to a surface at a point is that plane containing the tangents to curves which lie in the surface and which pass through the point.
2. Tangent plane normal. The gradient is normal to the tangent plane of a level surface. You should definitely know this fact. The equation of the tangent plane at $\underline{r}_0$ to the surface f = constant is $\underline{\nabla}f(\underline{r}_0)\cdot(\underline{r} - \underline{r}_0) = 0$. Compare this equation with the equation of the plane on p. 672.

3. Implicit differentiation. You should either learn the simple derivation just below the box on p. 810 or remember the formula $dy/dx = -(\partial z/\partial x)/(\partial z/\partial y)$. It looks almost like division of fractions, except for the minus sign. Don't forget the minus sign.

4. Related rates. If we know that $F(x,y) = \text{constant}$, then we can differentiate by the chain rule to get $(\partial F/\partial x)(dx/dt) + (\partial F/\partial y)(dy/dt) = 0$. The method is analogous to that used in Section 2.4.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

$f(0,0,1) = 1$, so the level surface is $x^2 + y^2 + z^2 = 1$, which is the unit sphere. $f_x = 2x$; $f_y = 2y$; $f_z = 2z$, so $\nabla f(0,0,1) = 2\mathbf{k}$.

5. $\nabla f(\mathbf{r}_0)$ is a normal to the surface at $\mathbf{r}_0$. $f_x = yz$; $f_y = xz$; $f_z = xy$, so $\nabla f(1,1,8) = 8\mathbf{i} + 8\mathbf{j} + \mathbf{k}$. $\|\nabla f(1,1,8)\| = \sqrt{129}$, so a unit normal is $(1/\sqrt{129})(8\mathbf{i} + 8\mathbf{j} + \mathbf{k})$.

9. By Example 4, Section 16.1, $\nabla(1/r) = -(\mathbf{r}/r^3)$. Therefore, we can choose $V = Qq/r$ to give $\mathbf{F} = -\nabla V$.

13. The equation of the tangent plane is $f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0) \times (y - y_0) + f_z(x_0,y_0,z_0)(z - z_0) = 0$. $\nabla f(x,y,z) = (2x + 3z,4y,3x)$, which is $(3,8,3)$ at $(1,2,1/3)$. Thus, the tangent plane is $3(x - 1) + 8(y - 2) + 3(z - 1/3) = 0$ or $3x + 8y + 3z = 20$.

17. The equation of the tangent line at (x_0,y_0) to the curve $f(x,y) = c$ is $f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) = 0$. $\underline{\nabla} f(x,y) = (2x,4y)$. At $(1,1)$, it becomes $(2,4)$. Thus, the equation for the tangent line is $2(x - 1) + 4(y - 1) = 0$ or $2x + 4y - 6 = 0 = x + 2y - 3$.

21. The normal to the tangent plane at $\underline{r}_0 = (x_0,y_0,z_0)$ of the level surface $f(x,y,z) = c$ is $\underline{\nabla} f(\underline{r}_0)$, so the normal line is $\underline{r}_0 + t\underline{\nabla} f(\underline{r}_0)$. $\underline{\nabla} f(x,y,z) = (-2x \exp(-x^2 - y^2 - z^2)$, $-2y \exp(-x^2 - y^2 - z^2)$, $-2z \exp(-x^2 - y^2 - z^2))$. Since $f_x = f_y = f_z$ at $(1,1,1)$, the normal line is $(1,1,1) + t(1,1,1)$.

25. Rewrite the given equation as $z = F(x,y) = 0$. Then $dy/dx = -(\partial z/\partial x)/(\partial z/\partial y)$. Let $z = x^2 + 2y^2 - 3$. Then $\partial z/\partial x = 2x$ and $\partial z/\partial y = 4y$; therefore, $dy/dx = -2x/4y = -x/2y$.

29. Using the method of Exercise 25, we let $z = x^3 - \sin y + y^4 - 4$. Then $\partial z/\partial x = 3x^2$ and $\partial z/\partial y = -\cos y + 4y^3$; therefore, $dy/dx = 3x^2/(\cos y - 4y^3)$.

33. Let $z = \cos(x + y) - x - 1/2$. Then $\partial z/\partial x = -\sin(x + y) - 1$ and $\partial z/\partial y = -\sin(x + y)$. At $(0,\pi/3)$, $\partial z/\partial x = -\sqrt{3}/2 - 1$ and $\partial z/\partial y = -\sqrt{3}/2$; therefore, $dy/dx = (\sqrt{3}/2 + 1)/(-\sqrt{3}/2) = -1 - 2\sqrt{3}/3$.

37. $\partial z/\partial y = -5y^4$, which is 0 at $(0,0)$. At that point, the graph of $x = y^5$ is pointed, so the slope is infinite.

41. Let $F(x,y) = x^4 + y^4 - 1$. Then $\partial F/\partial x = 4x^3$ and $\partial F/\partial y = 4y^3$; therefore, formula (2) gives $4x^3(dx/dt) + 4y^3(dy/dt) = 0$ or $x^3(dx/dt) + y^3(dy/dt) = 0$.

45. Let $F(x,y) = \ln(x \cos y) - x = \ln x + \ln \cos y - x$. Then $\partial F/\partial x = 1/x - 1$ and $\partial F/\partial y = -\sin y/\cos y = -\tan y$. Therefore, formula (2) gives $(1/x - 1)(dx/dt) + (-\tan y)(dy/dt) = 0$.

49. The direction normal to the surface is $\underline{\nabla}f(1,1,\sqrt{3})$. We find $f_x(1,1,\sqrt{3}) = 2x|_{(1,1,\sqrt{3})} = 2$; $f_y(1,1,\sqrt{3}) = 2y|_{(1,1,\sqrt{3})} = 2$; and $f_z(1,1,\sqrt{3}) = -2z|_{(1,1,\sqrt{3})} = -2\sqrt{3}$, so $\underline{\nabla}f(1,1,\sqrt{3}) = (2,2,-2\sqrt{3})$. Let $\underline{v} = a(1,1,-\sqrt{3})$ denote the velocity of the particle. Then its speed $s = \|\underline{v}\| = 10 = a\sqrt{1+1+3} = \sqrt{5}a$, so $a = 2\sqrt{5}$. Hence, $\underline{v} = 2\sqrt{5}(1,1,-\sqrt{3})$. It crosses the xy-plane when $z = 0 = \sqrt{3} - 2\sqrt{3}\sqrt{5}t$; i.e., when $t = 1/2\sqrt{5} = \sqrt{5}/10$. Then $x = 1 + 2\sqrt{5}/2\sqrt{5} = 2 = y$, so it crosses at $(2,2,0)$, which occurs $\sqrt{5}/10$ seconds later.

SECTION QUIZ

1. Suppose that $2x + 3y + x^2y^2 = 0$ and $y = f(x)$, then $f'(x)$ is <u>not</u> $\dfrac{-2+2xy^2}{3+2x^2y}$. What is the common mistake made here which might cost points on an exam?

2. Find $f'(x)$ if $y = f(x)$ is a function of x :

   (a) $y \cos x - \sin(e^y) + x$

   (b) $e^{xy} - y \ln xy$

3. Use the gradient to find the tangent plane to the surface $x^2y^2/2 + 3xy = \sqrt{x} - \cos y + z$ at $(1,0,-3/2)$.

4. If $z = F(x,y)$ and x and y are both functions of t , find a relationship between dx/dt and dy/dt when $xy - z + 2 = 0$.

5. A hungry little bear cub is looking for honey. When he eyes a beehive which has a paraboloid shape given by $z = 16 - x^2 - y^2$, he quickly approaches, expecting a delicious breakfast. However, at $(1,2)$, he is met by an angry swarm of bees. The stinging causes the cub to jump in a direction normal to the tangent plane of the beehive at $(1,2)$.

   (a) What is the direction in which the bear cub springs?

   (b) What is the tangent plane of the beehive at $(1,2)$?

ANSWERS TO PREREQUISITE QUIZ

1. $z = 2y + 1$

2. $z = \pi x - y + \pi/2$

3. $2y\underline{i} + (2x - \sin y)\underline{j}$

4. $6x\underline{i} + z\underline{j} + (y - 3z^2)\underline{k}$

5. $x/(1 - y)$

ANSWERS TO SECTION QUIZ

1. The correct answer is $-\left(\frac{2 + 2xy^2}{3 + 2x^2y}\right)$. Note parentheses.

2. (a) $(y \sin x - 1)/(\cos x - e^y\cos(e^y))$

   (b) $(y/x - ye^{xy})/(xe^{xy} - \ln xy - 1)$

3. $z = -x/2 + 3y - 1$

4. $dx/dt = -(x/y)dy/dt$

5. (a) $-2\underline{i} - 4\underline{j} - \underline{k}$

   (b) $z = 21 - 2x - 4y$

16.3 Maxima and Minima

PREREQUISITES

1. Recall how to find the local extrema of a function of one variable (Section 3.5).
2. Recall how to classify the local extrema of a function of one variable by using the first and second derivative tests (Sections 3.2 and 3.3).

PREREQUISITE QUIZ

1. Find the critical points of $f(x) = x^3 + 8x^2 + 5$.
2. If f is continuously differentiable, $f'(x_0) = 0$, and x_0 is a local maximum point, what is the sign of $f'(x)$ near x_0 for $x < x_0$ and $x > x_0$?
3. If x_0 is a local minimum point and $f''(x_0) \neq 0$, what is the sign of $f''(x_0)$?
4. Classify the critical points of $f(x) = x^3 - x$.

GOALS

1. Be able to find the extrema of a two-variable function and classify them as local minima, local maxima, or saddle points.

STUDY HINTS

1. Definitions. A point is a local minimum if all other points nearby have larger values. Notice that this definition is local, referring to nearby points. Compare this definition with that in one-variable calculus (Section 3.2). A global minimum of f is the smallest value taken by f on its entire domain. Similar statements may be made for maxima. An extremum (local or global) is either a minimum or maximum.

2. Critical points. For functions of two variables, a critical point is any point where both partials vanish (i.e., equal zero). All local extrema are critical points, but not all critical points are local extrema. For example, saddle points are critical points, but not local extrema. Example 6 shows an interesting problem in which the critical point is not an extremum.
3. Validity of minimizing squares. In Example 4, we want to solve $\partial d/\partial x = 0$ and $\partial d/\partial y = 0$. By differentiating d^2, the chain rule gives $2d(\partial d/\partial x)$ and $2d(\partial d/\partial y)$, so we again are solving $\partial d/\partial x = 0$ and $\partial d/\partial y = 0$ since $d > 0$.
4. Maximum-minimum test. Consider $Ax^2 + 2Bxy + Cy^2$. If $AC - B^2 < 0$, the origin is not a local extremum. If $AC - B^2 > 0$ and $A > 0$, we have a local minimum at $(0,0)$. If $AC - B^2 > 0$ and $A < 0$, $(0,0)$ is a local maximum. This should be memorized. Now, if we let $A = f_{xx}$, $B = f_{xy}$, and $C = f_{yy}$, we get the second derivative test on p. 817.
5. Proving the max-min test. In step 1 of the proof of the maximum-minimum test for quadratic functions, it is stated that if $AC - B^2 > 0$, $A \neq 0$. If $A = 0$, we have $-B^2 > 0$, which is impossible.
6. Finding extrema. If $\partial f/\partial x = 0$ or $\partial f/\partial y = 0$ have more than one solution, each combination of (x,y) must be considered. You must be complete in your analysis. See Example 10.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Notice that the denominator is smallest when $y = 0$; it has no largest value, so we expect the minimum and maximum to occur when $y = 0$. This should be at a critical point of $x^3 - 3x$, i.e., where $3x^2 - 3 = 0$ or $x = \pm 1$. From the graph, we see that the minimum is at $(1,0)$ and the maximum is at $(-1,0)$.

5. The critical points occur where $\partial f/\partial x$ and $\partial f/\partial y$ both vanish. $\partial f/\partial x = (-2x)\exp(-x^2 - 7y^2 + 3)$ and $\partial f/\partial y = (-14y)\exp(-x^2 - 7y^2 + 3)$. The partial derivatives vanish at $(0,0)$, so the origin is a critical point. It is a local maximum since $-x^2 - 7y^2 \leqslant 0$.

9. Let x and y be the length of the sides of the base. Then the height must be $256/xy$. The cost of the box is $C(x,y) = bxy + 2sxz + 2syz$, i.e., $A(x,y) = bxy + 512s/y + 512s/x$. The partial derivatives $\partial C/\partial x = by - (512/x^2)s$ and $\partial C/\partial y = bx - (512/y^2)s$ vanish at $(8\cdot\sqrt[3]{s/b}, 8\cdot\sqrt[3]{s/b})$, so we have a square based box and the height is $256/xy = 4/\sqrt[3]{s^2/b^2}$ $= 4b^{2/3}/s^{2/3}$.

13. We have $A = -1$, $B = 3/2$, and $C = 1$, so $AC - B^2 = -13/4 < 0$. Therefore, $(0,0)$ is a saddle point.

17. The critical points occur where $\partial f/\partial x = 0 = \partial f/\partial y$. Classify them by using the second derivative test. Here, $f_x = 2x + 6$ and $f_y = 2y - 4$, so the critical point is at $(-3,2)$. Further, $A = f_{xx} = 2$, $B = f_{xy} = 0$, and $C = f_{yy} = 2$. Therefore, $AC - B^2 = 4 > 0$ and $A > 0$, so $(-3,2)$ is a local minimum.

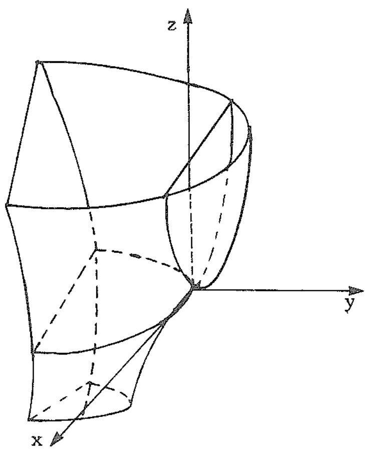
21. $f_x = 2x - 6 = 0$ when $x = 3$. $f_y = 2y - 14 = 0$ when $y = 7$. Therefore, $(3,7)$ is the critical point. $A = f_{xx} = 2 = f_{yy} = C$ and $B = f_{xy} = 0$, so $AC - B^2 = 4 > 0$. Since $f_{xx} > 0$, the second derivative test tells us that $(3,7)$ is a local minimum.

25. We have $f_x = 6x + 2y - 3$, $f_y = 2x + 4y + 2$; therefore, $B = f_{xy} = 2$, $A = f_{xx} = 6$, and $C = f_{yy} = 4$. Solving $f_x = f_y = 0$, we find the critical point $(4/5,-9/10)$. Since $A > 0$ and $AC - B^2 > 0$, $(4/5,-9/10)$ is a local minimum.

29. $f_x = y\cos xy/(2 + \sin xy) = 0$ implies $y = 0$ or $xy = \pi/2 + n\pi$, where n is any integer. Also $f_y = x\cos xy/(2 + \sin xy) = 0$ implies $x = 0$ or $xy = \pi/2 + n\pi$. Hence $(0,0)$ and all points satisfying $xy = \pi/2 + n\pi$ are critical points.

29. (continued)

$A = f_{xx} = [-y^2 \sin xy(2 + \sin xy) - (y \cos xy)^2]/(2 + \sin xy)^2 = -y^2(2 \sin xy + \sin^2 xy + \cos^2 xy)/(2 + \sin xy)^2 = -y^2(1 + 2 \sin xy)/(2 + \sin xy)^2$. Similarly, by symmetry, $C = f_{yy} = -x^2(1 + 2 \sin xy)/(2 + \sin xy)^2$. $B = f_{xy} = [(\cos xy - xy \sin xy)(2 + \sin xy) - xy \cos^2 xy]/(2 + \sin xy)^2 = (2 \cos xy - 2xy \sin xy + \cos xy \sin xy - xy \sin^2 xy - xy \cos^2 xy)/(2 + \sin xy)^2 = (2 \cos xy - 2xy \sin xy + \cos xy \sin xy - xy)/(2 + \sin xy)^2$. If $(x,y) = (0,0)$, we get $A = 0 = C$ and $B = 1/2$, so $AC - B^2 = -1/4 < 0$. Thus $(0,0)$ is a saddle point.

33. 

$f_x = 2x = 0$ at $(0,0)$. $f_y = 3y^2 = 0$ at $(0,0)$. $f_{xx} = 2$; $f_{yy} = 6y$; and $f_{xy} = 0$. Hence, $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$, meaning the second derivative test fails. Plot the graph. When $y = 0$, the level curve is the parabola $z = x^2$. When $x = 0$, the level curve is the cubic $z = y^3$, which has an inflection point at $(0,0)$. When $z = 0$, the level curve is described by $y = -x^{2/3}$. Hence, $(0,0)$ must be a saddle point.

37. (a) Minimize w by setting $\partial w/\partial p_i = 0$. Since $w = c_1 y + c_2$, $\partial w/\partial p_i = c_1(\partial y/\partial p_i) = c_1(\partial/\partial p_i)(\Sigma T_i(p_i/p_{i+1})^{(n-1)/n})$. Since only two terms involve p_i , we have $c_1(\partial/\partial p_i)(T_{i-1}(p_{i-1}/p_i)^{(n-1)/n} + T_i(p_i/p_{i+1})^{(n-1)/n}) = c_1(\partial/\partial p_i)(T_{i-1}p_{i-1}^{1-(1/n)}p_i^{(1/n)-1} + T_i p_i^{1-(1/n)}p_{i+1}^{(1/n)-1}) = c_1[T_{i-1}p_{i-1}^{1-(1/n)}(1/n - 1)p_i^{(1/n)-2} + T_i p_{i+1}^{(1/n)-1}(1 - 1/n)p_i^{-1/n}] = c_1(1 - 1/n)[T_i p_{i+1}^{(1/n)-1}p_i^{1/n} - T_{i-1}p_{i-1}^{1-(1/n)}p_i^{(1/n)-2}]$. Hence,

37. (a) (continued)

$\partial w/\partial p_i = 0$ if $T_i p_{i+1}^{(1/n)-1} p_i^{-1/n} = T_{i-1} p_{i-1}^{1-(1/n)} p_i^{(1/n)-2}$. Solve for $T_{i-1}/T_i = p_{i+1}^{(1/n)-1} p_i^{-1/n} p_{i-1}^{(1/n)-1} p_i^{2-(1/n)} = p_{i+1}^{(1/n)-1} p_i^{2-(2/n)} \times p_{i-1}^{(1/n)-1} = (p_i^2/(p_{i-1}p_{i+1}))^{1-(1/n)}$, which is the necessary condition to minimize w.

(b) $y = T_0(p_0/p_1)^{1-(1/n)} + T_1(p_1/p_2)^{1-(1/n)} + K$, where $K = T_2 = (p_2/p_3)^{1-(1/n)} + T_3(p_3/p_4)^{1-(1/n)}$. Then $y = T_0 p_0^{1-(1/n)} p_1^{(1/n)-1} + T_1 p_1^{1-(1/n)} p_2^{(1/n)-1} + K$. Consequently, $\partial w/\partial p_1 = c_1 [T_0 p_0^{1-(1/n)} \times (1/n - 1) p_1^{(1/n)-2} + T_1 p_2^{(1/n)-1}(1 - 1/n) p_1^{-1/n}] = c_1(1 - 1/n) \times [T_1 p_2^{(1/n)-1} p_1^{-1/n} - T_0 p_0^{1-(1/n)} p_1^{(1/n)-2}]$. Hence, $\partial w/\partial p_1 = 0$ if $T_1 p_2^{(1/n)-1} p_1^{-1/n} = T_0 p_0^{1-(1/n)} p_1^{(1/n)-2}$. Similarly, $\partial w/\partial p_2 = 0$ if $T_1 = T_2 p_3^{(1/n)-1} p_2^{2-(2/n)} p_1^{(1/n)-1}$; and $\partial w/\partial p_3 = 0$ if $T_2 = T_3 p_4^{(1/n)-1} p_3^{2-(2/n)} p_2^{(1/n)-1}$. From $\partial w/\partial p_1 = 0$, $p_1^{2-(2/n)} = (T_0/T_1) p_2^{1-(1/n)} p_0^{1-(1/n)}$, so $p_1 = (T_0/T_1)^{n/(2n-2)} \sqrt{p_0 p_2}$. Similarly, $p_2 - (T_1/T_2)^{n/(2n-2)} \sqrt{p_1 p_3}$ and $p_3 = (T_2/T_3)^{n/(2n-2)} \times \sqrt{p_2 p_4}$. Substitute for p_3 into p_2 to get $(T_1/T_2)^{n/(2n-2)} \times \sqrt{p_1}(T_2/T_3)^{n/(4n-4)} \sqrt{p_2 p_4}$. Hence, $p_2^{3/4} = T_1^{n/(2n-2)} T_2^{-n/(4n-4)} \times T_3^{-n/(4n-4)} \sqrt{p_1} \sqrt[4]{p_4}$, so $p_2 = (T_1^2/T_2T_3)^{n/(3n-3)} p_1^{2/3} p_4^{1/3}$. Substitute this into p_1 to get $(T_0/T_1)^{n/(2n-2)} \sqrt{p_0} (T_1^2/T_2T_3)^{n/(6n-6)} \times p_1^{1/3} p_4^{1/6}$. Hence, $p_1^{2/3} = \sqrt{p_0}\, p_4^{1/6} T_0^{n/(2n-2)} T_1^{-n/(6n-6)} (T_2T_3)^{-n/(6n-6)}$. Thus, $p_1 = p_0^{3/4} p_4^{1/4} T_0^{3n/(4n-4)} T_1^{-n/(4n-4)} (T_2T_3)^{-n/(4n-4)} = [p_0^3 p_4 (T_0^3/T_1T_2T_3)^{n/(n-1)}]^{1/4}$. Substitute this into p_2 to get $(T_1^2/T_2T_3)^{n/(3n-3)} p_4^{1/3} \cdot [p_0^3 p_4 (T_0^3/T_1T_2T_3)^{n/(n-1)}]^{1/6} = p_0^{1/2} p_4^{1/2} T_0^{n/(2n-2)} T_1^{n/(2n-2)} T_2^{-n/(2n-2)} T_3^{-n/(2n-2)} = [p_0 p_4 (T_0T_1/T_2T_3)^{n/(n-1)}]^{1/2}$. Substitute this into p_3 to get $p_3 = T_2^{n/(2n-2)} \times T_3^{-n/(2n-2)} p_4^{1/2} \cdot [p_0 p_4 (T_0T_1/T_2T_3)^{n/(n-1)}]^{1/4} = p_0^{1/4} p_4^{3/4} (T_0T_1)^{n/(4n-4)} T_2^{n/(4n-4)} T_3^{-3n/(4n-4)} = [p_0 p_4^3 (T_0T_1T_2/T_3^3)^{n/(n-1)}]^{1/4}$.

41. (a) $f_x = 2x + 3y$ and $f_y = -2y + 3x$. At $(0,0)$, $f_x = f_y = 0$. $f_{xx} = 2$; $f_{yy} = -2$; and $f_{xy} = 3$. Hence $f_{xx}f_{yy} - f_{xy}^2 = -4 - 9 = -13 < 0$. The second derivative test tells us that $(0,0)$ is a saddle point.

(b) $f_x = 2x + Cy$ and $f_y = 2y + Cx$. At $(0,0)$, $f_x = f_y = 0$. $f_{xx} = 2$; $f_{yy} = 2$; and $f_{xy} = C$, so $f_{xx}f_{yy} - f_{xy}^2 = 4 - C^2$. Hence if $-2 < C < 2$, $(0,0)$ is a strict minimum; but if $C < -2$ or $C > 2$, then $(0,0)$ is a saddle point. If $C = \pm 2$, then $f(x) = (x \pm y)^2$, so $(0,0)$ is a minimum. The behavior changes qualitatively at $C = \pm 2$.

45. $s(m,b) = \Sigma(y_i - mx_i - b)^2$. Therefore $\partial s/\partial b = \Sigma(-2)(y_i - mx_i - b)$. If $\partial s/\partial b = 0$, then $\Sigma(y_i - mx_i - b) = 0$. Using the rules for summation (See Section 4.1), $\Sigma y_i - m\Sigma x_i - nb = 0$, so $m\Sigma x_i + nb = \Sigma y_i$. Also, $\partial s/\partial m = \Sigma(-2x_i)(y_i - mx_i - b)$. If $\partial s/\partial m = 0$, then $\Sigma x_i(y_i - mx_i - b) = 0$, so $\Sigma x_i y_i = m\Sigma x_i^2 + b\Sigma x_i$.

49. (a) The test gives $g(x,y) = Ax^2 + 2Bxy + Cy^2 = A[x^2 + 2Bxy/A + Cy^2/A] = A[x^2 + 2Bxy/A + (By/A)^2 - (By/A)^2 + Cy^2/A] = A[(x + By/A)^2 - y^2(B^2 - CA)/A^2]$. Let $e = \sqrt{B^2 - CA}/A$. Then $g(x,y) = A[(x + By/A)^2 - y^2e^2] = A[(x + (By/A) - ey)(x + (By/A) + ey)]$.

(b) $g(x,y) = 0$ if $x + By/A - ey = 0$ or if $x + By/A + ey = 0$. Solve these for y as follows: $x + y(B/A - e) = 0$ implies $y = x/(e - B/A) = Ax/(Ae - B)$; and $x + y(B/A + e) = 0$ implies $y = -x/(B/A + e) = -Ax/(B + Ae)$. These are lines that intersect at the origin.

(c) $g(x,y)$ is positive if $[x + By/A - ey > 0$ and $x + By/A + ey > 0]$ or $[x + By/A - ey < 0$ and $x + By/A + ey < 0]$. This reduces to $[y > Ax/(Ae - B)$ and $y > -Ax/(B + Ae)]$ or $[y < Ax/(Ae - B)$ and $y < -Ax/(B + Ae)]$. These are the regions above both lines and below both lines, respectively.

49. (d) If $A = B = 0$, then $AC - B^2 = 0$ and the test does not apply. Rewrite $g(x,y)$ as $y(2Bx + Cy)$. Note that $y = 0$ and $2Bx + Cy = 0$ are two lines intersecting at the origin. Thus, $g(x)$ is positive in the region above $2Bx + Cy = 0$ and the negative x-axis. Also, $g(x)$ is positive in the region below $2Bx + Cy = 0$ and the positive x-axis.

53. As in Example 10, we minimize the square of the distance: $f(x,y) = x^2 + y^2 + (4x^2 + y^2 - a)^2$. The partials are $f_x(x,y) = 2x + 2(4x^2 + y^2 - a)8x = 2x(32x^2 + 8y^2 - 8a + 1)$ and $f_y(x,y) = 2y + 2(4x^2 + y^2 - a)2y = 2y(8x^2 + 2y^2 - 2a + 1)$. Thus, we need $x = 0$ or $32x^2 + 8y^2 - 8a + 1 = 0$. Also, we need $y = 0$ or $8x^2 + 2y^2 - 2a + 1 = 0$. Again, there are four possibilities. $(0,0)$ is one critical point.

For $x = 0$ and $8x^2 + 2y^2 - 2a + 1 = 0$, we have $2y^2 - 2a + 1 = 0$ or $y = \pm\sqrt{(2a - 1)/2}$.

For $y = 0$ and $32x^2 + 8y^2 - 8a + 1 = 0$, we have $32x^2 - 8a + 1 = 0$ or $x = \pm\sqrt{(8a - 1)/32}$.

For $32x^2 + 8y^2 - 8a + 1 = 0$ and $8x^2 + 2y^2 - 2a + 1 = 0$, subtracting 4 times the second equation from the first gives $-3 = 0$, which is impossible.

The second derivatives are $f_{xx}(x,y) = 2(32x^2 + 8y^2 - 8a + 1) + 2x(64x) = 192x^2 + 16y^2 - 16a + 2$, and $f_{yy}(x,y) = 2(8x^2 + 2y^2 - 2a + 1) + 2y(4y) = 16x^2 + 12y^2 - 4a + 2$, and $f_{xy}(x,y) = 32xy$.

At $(x,y) = (0,0)$, we have $f_{xx} = -16a + 2$, $f_{yy} = -4a + 2$, and $f_{xy} = 0$. Thus, $f_{xx}f_{yy} - f_{xy}^2 = 64a^2 - 40a + 4$. At $(x,y) = (0,\pm\sqrt{(2a - 1)/2})$, we have $f_{xx} = -6$, $f_{yy} = 8a - 4$, and $f_{xy} = 0$.

53. (continued)

Thus, $f_{xx}f_{yy} - f_{xy}^2 = -48a + 24$. At $(x,y) = (\pm\sqrt{(8a - 1)/32},0)$, we have $f_{xx} = 32a - 4$, $f_{yy} = 3/2$, and $f_{xy} = 0$. Thus, $f_{xx}f_{yy} - f_{xy}^2 = 48a - 6 = 6(8a - 1)$.

Therefore, if $a < 1/8, (0,0)$ is the only critical point, $f_{xx} > 0$, and $f_{xx}f_{yy} - f_{xy}^2 > 0$, so it is a local minimum. If $a = 1/8$, the critical point is $(0,0)$ and it is a local minimum. If $1/8 < a < 1/2$, then the critical points are $(0,0)$ and $(\pm\sqrt{(8a - 1)/32},0)$. In this case, the second derivative test tells us that $(\pm\sqrt{(8a - 1)/32},0)$ gives the minimum. If $a = 1/2$, the second derivative test fails for $(0,0)$ and tells us that $(\pm\sqrt{3/32},0)$ is a local minimum. Substituting back into $f(x,y)$ tells us it is the global minimum. If $a > 1/2$, all three points are critical points and the second derivative test still tells us that $(\pm\sqrt{(8a - 1)/32},0)$ is the minimum. Thus, $(0,0,0)$ is the closest point if $a \leqslant 1/8$ and $(\pm\sqrt{(8a - 1)/32},0),(8a - 1)/8)$ are the closest points if $a \geqslant 1/8$.

SECTION QUIZ

1. Consider a two-variable function, $f(x,y)$. Explain how $f_{xx}f_{yy} - [f_{xy}]^2$ is used to classify critical points of f .

2. Find the critical points of $f(x,y)$ and classify them:

   (a) $f(x,y) = x^2 - 5xy + 8y^2$

   (b) $f(x,y) = e^{xy} + 3x - 3y$ [Hint: A calculator and a graph may be useful.]

   (c) $f(x,y) = xy - x^2 + y^2$

   (d) $f(x,y) = 1 - 2y^2$

3. Find and classify all critical points of $z = (y^3/3 - y)(x^2/2 - 2)$.

4. A sadistic dentist likes to explain to his patients how they can expect to feel pain. He explains that pain is a function of c , the dental bill which the patient pays without the help of insurance; t , the time spent by the patient trying to understand the dentist's pain formula; and n , the number of teeth to be yanked out. This dentist points out that his patient's pain, P , is given by $P = c^2t + 2^n$. Explain why P has no critical points. (Assume that P is a continuous function of n because this dentist sometimes mistakenly pulls out only parts of a tooth.)

ANSWERS TO PREREQUISITE QUIZ

1. $-16/3$, 0

2. $f'(x_0) < 0$ for $x < x_0$ and x near x_0 ; $f'(x_0) > 0$ for $x > x_0$ and x near x_0 .

3. Positive

4. $-\sqrt{1/3}$ = local maximum; $+\sqrt{1/3}$ = local minimum.

ANSWERS TO SECTION QUIZ

1. Let (x_0,y_0) be a critical point and let $A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, and $C = f_{yy}(x_0,y_0)$. If $AC - B^2 > 0$ and $A > 0$, then (x_0,y_0) is a local minimum. If $AC - B^2 > 0$ and $A < 0$, then (x_0,y_0) is a local maximum. If $AC - B^2 < 0$, then (x_0,y_0) is a saddle point. If $AC - B^2 = 0$, the test is inconclusive.

2. (a) $(0,0)$ is a local minimum; $f(x,y) = (x + 5y/2)^2 + 7y^2/4$.

   (b) There are no critical points.

   (c) $(0,0)$ is a saddle point.

   (d) $(x,0)$ is a local maximum.

3. $(0,1)$ is a local maximum; $(0,-1)$ is a local minimum; $(\pm 2,0)$, $(\pm 2,\sqrt{3})$ and $(\pm 2,-\sqrt{3})$ are all saddle points.

4. $\partial P/\partial n = 2^n(\ln 2) \neq 0$ for any n. A critical point must satisfy $\partial P/\partial c = \partial P/\partial t = \partial P/\partial n = 0$.

16.4 Constrained Extrema and Lagrange Multipliers

PREREQUISITES

1. Recall the extreme value theorem (Section 3.5)
2. Recall how to find the local extrema of a function of several variables (Section 16.3).
3. Recall how to solve minimum-maximum word problems (Section 3.5).

PREREQUISITE QUIZ

1. Find the minimum and maximum of $f(x) = x^2$ on $[-1,3]$.
2. If 12 meters of rope are available to rope off a region, what is the maximum area of the region?
3. State the extreme value theorem.
4. If $f(x,y)$ is a function of two variables, what must be the value of $\partial f/\partial x$ and $\partial f/\partial y$ at a local extreme point?

GOALS

1. Be able to use the method of Lagrange multipliers to find the minimum and maximum for functions with constraints.
2. Be able to find the minimum and maximum of a two-variable function on a region with a boundary.

STUDY HINTS

1. Constraints. In Chapter 3, we limited our analysis to parts of the real line by specifying endpoints. Analogously, a boundary curve limits the region of our analysis in the plane.

2. Locating boundary extrema. Two methods are used. The first is to parametrize the given boundary and then differentiate to determine the local minimum and local maximum points. See Step 2 of Example 1. The second method is that of Lagrange multipliers.

3. Lagrange multipliers. Rather than remembering the equations (1), there is another way to get the same result. We transform the constraint $g(x,y) = c$ into $g(x,y) - c = 0$. Being sure that the constraint equals zero, we look for critical points of $f(x,y) - \lambda[g(x,y) - c]$, i.e., the original function minus λ times the constraint.

4. Extrema on a region. The two methods discussed above only locate extrema on a boundary. Don't forget to analyze the critical points inside the region.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the method of Example 1. $\partial f/\partial x = 4x$ and $\partial f/\partial y = 6y$, so $(0,0)$ is the only critical point. On the boundary, we have $h(t) = 2\cos^2 t + 3\sin^2 t$. Thus, $h'(t) = -4\cos t \sin t + 6 \sin t \cos t = 2\sin t\cos t = \sin 2t$. The possible extrema points may occur at $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. At each of these points, $(\cos t, \sin t) = (\pm\sqrt{2}/2, \pm\sqrt{2}/2)$. Thus, the minimum value of 0 occurs at $(0,0)$ and the maximum occurs at four boundary points; the maximum value is $5/2$.

5. Use the method of Lagrange multipliers. We want to analyze $k(x,y,\lambda) = 3x + 2y + \lambda(2x^2 + 3y^2 - 3)$. $k_x = 3 + 4x\lambda$, $k_y = 2 + 6y\lambda$, and $k_\lambda = 2x^2 + 3y^2 - 3$. Setting the partials equal to 0, we get $x = -3/4\lambda$, $y = -1/3\lambda$; therefore, $k_\lambda = 9/8\lambda^2 + 1/3\lambda^2 - 3 = 0 = 35/24\lambda^2 - 3$. Thus, $\lambda = \pm\sqrt{35/72}$, $x = \mp 9/\sqrt{70}$, and $y = \mp\sqrt{8/35}$. Since $f_x = 3$ and $f_y = 2$, there are no critical points. $f(9/\sqrt{70}, \sqrt{8/35}) = \sqrt{35/2}$ and

5. (continued)

$f(-9/\sqrt{70},-\sqrt{8/35}) = -\sqrt{35/2}$, which are the extreme values.

9. We want to analyze $k(x,y,\lambda) = xy + \lambda(x + y - 1)$. $k_x = y + \lambda$; $k_y = x + \lambda$; $k_\lambda = x + y - 1$. $k_x = k_y = k_\lambda = 0$ implies $y = -\lambda = x$, so $-2\lambda - 1 = 0$; i.e., $\lambda = -1/2$. Then $x = y = 1/2$. $f(1/2,1/2) = 1/4$. Hence, $(1/2,1/2)$ is the extreme point. Consider $f(0,1) = 0$. Since $(1/2,1/2)$ is an extreme point and $f(0,1) < f(1/2,1/2)$, the point $(1/2,1/2)$ must be the maximum point, with value 1/4 .

13. Use Lagrange multipliers. Let $f(x,y,z,\lambda) = 8xyz^2 - 200{,}000(x + y + z) + \lambda(x + y + z - 100{,}000)$. Then $f_x = 8yz^2 - 200{,}000 + \lambda$; $f_y = 8xz^2 - 200{,}000 + \lambda$; $f_z = 16xyz - 200{,}000 + \lambda$; $f_\lambda = x + y + z - 100{,}000$. Since $f_x = 0$, $\lambda = 200{,}000 - 8yz^2$. Since $f_y = 0$, $\lambda = 200{,}000 - 8xz^2$. Therefore $x = y$. Since $f_z = 0$, $\lambda = 200{,}000 - 16xyz = 200{,}000 - 16x^2z$. Therefore, $8xz^2 = 16x^2z$, so $z = 2x$ (since $x \neq 0$, $z \neq 0$). Substitute for y and z in $f_\lambda = 0$, giving $4x = 100{,}000$. Hence $x = y = 25{,}000$ and $z = 50{,}000$.

17. Let $k(D_1,D_2,\lambda) = \ell_1(a + bD_1) + \ell_2(a + bD_2) + \lambda(c\ell_1Q_1^2/D_1^5 + c\ell_2Q_2^2/D_2^5 - h)$. Then $\partial k/\partial D_1 = \ell_1 b - 5\lambda c\ell_1Q_1^2/D_1^b = \ell_1(b - 5\lambda cQ_1^2/D_1^6)$; $\partial k/\partial D_2 = \ell_2(b - 5\lambda cQ_2^2/D_2^6)$; and $\partial k/\partial\lambda = c\ell_1Q_1^2/D_1^5 + c\ell_2Q_2^2/D_2^5 - h$. If $\partial k/\partial D_1 = 0$, then $\lambda = bD_1^6/5cQ_1^2$. If $\partial k/\partial D_2 = 0$, then $\lambda = bD_2^6/5cQ_2^2$. Equate these to get $D_2/D_1 = (Q_2/Q_1)^{1/3}$.

21. (a) $n_1i_1^2/q_1 = n_1^2i_1^2/\alpha xh$. Since n_1i_1 is constant, so is $n_1^2i_1^2/\alpha h$. Call this constant K . Similarly, $n_2i_2^2/q^2 = K/y$. Thus, C simplifies to $\rho\pi K[(D_1 + x)/x + (D_2 - y)/y] = \rho\pi K(D_1/x + D_2/y)$.

(b) To minimize C , we look at $f(x,y,\lambda) = \rho\pi K(D_1/x + D_2/y) + \lambda(x + y - (1/2)(D_2 - D_1))$. $f_x = -\rho\pi KD_1/x^2 + \lambda$, $f_y = -\rho\pi KD_2/y^2 + \lambda$ and $f_\lambda = x + y - (1/2)(D_2 - D_1)$. $f_x = f_y = 0$ implies $D_1/x^2 =$

21. (b) (continued)

D_2/y^2 , so $y = \sqrt{D_2/D_1}x$. $f_\lambda = 0$ implies $(1 + \sqrt{D_2/D_1})x - (1/2)(D_2 - D_1) = 0$, so $x = (D_2 - D_1)/2(1 + \sqrt{D_2/D_1})$ and $y = (D_2 - D_1)\sqrt{D_2/D_1}/2(1 + \sqrt{D_2/D_1})$.

SECTION QUIZ

1. The function $f(x,y) = 1/xy + x + y$ has local extrema, yet it has no extreme values in the disk $x^2 + y^2 \leqslant 2$.
   (a) Why does this function with constraints not have extreme values?
   (b) Find the local extrema points.
2. The function xy^2 has no critical points in the first quadrant, but extreme values exist in the rectangular region $1 \leqslant x \leqslant 3$ and $3 \leqslant y \leqslant 4$. Find the extreme values.
3. Find the minimum and maximum of x^2y^2 in the region bounded by $y = x^2$, $y = -\sqrt{1 - x^2}$, and $x = \pm 1$.
4. A certain little gopher's weight gain depends on how much vegetables the Green family plants. Due to the shape of the terrain, the vegetables at point (x,y) provide $36 - x^2$ weight units for the gopher. The sprinkler system provides easy passage parallel to the x-axis only, so the weight gain is given by $W = 36 - x^2 - y$. All of the Greens' vegetables are planted in the disk $x^2 + y^2 \leqslant 4$.
   (a) Where does the gopher minimize his weight gain, W , if he eats the Greens' vegetables?
   (b) At what spot should the gopher eat to become as chubby as possible?

ANSWERS TO PREREQUISITE QUIZ

1. Minimum = 0 ; maximum = 9 .

2. $9\ m^2$

3. A minimum and maximum must exist if (i) the graph is continuous and (ii) the domain is closed.

4. $\partial f/\partial x = \partial f/\partial y = 0$

ANSWERS TO SECTION QUIZ

1. (a) It is not continuous at $(0,0)$.

   (b) $(1,1,3)$

2. Minimum = 3 ; maximum = 48 .

3. Minimum = 0 on x-axis $(-1 \leqslant x \leqslant 1)$ and on y-axis $(-1 \leqslant y \leqslant 0)$; maximum = 1 at $(\pm 1,1)$.

4. (a) $(-\sqrt{15}/2,1/2)$

   (b) $(0,-2)$

16.R Review Exercises for Chapter 16

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The gradient is $(\partial f/\partial x)\underline{i} + (\partial f/\partial y)\underline{j}$. $\underline{\nabla}f(x,y) = [y \exp(xy) - y \sin(xy)]\underline{i} + [x \exp(xy) - x \sin(xy)]\underline{j}$.

5. (a) Since $\underline{d}$ is a unit vector, the directional derivative is $\underline{\nabla}f \cdot \underline{d}$. $\underline{\nabla}f = 3x^2 \cos(x^3 - 2y^3)\underline{i} - 6y^2\cos(x^3 - 2y^3)\underline{j}$. At $(1,-1)$, $\underline{\nabla}f = 3 \cos(3)\underline{i} - 6 \cos(3)\underline{j}$; therefore, the directional derivative is $(3/\sqrt{2})\cos(3) + (6/\sqrt{2})\cos(3) = (9/\sqrt{2})\cos(3)$.

(b) The direction of fastest increase is $\underline{\nabla}f = 3 \cos(3)\underline{i} - 6 \cos(3)\underline{j}$ or $(\underline{i} - 2\underline{j})/\sqrt{5}$.

9. The normal to the tangent plane is $\underline{\nabla}f(x_0,y_0,z_0)$. Rewrite the surface as $x^3 + 2y^2 - z = 0$, so $\underline{\nabla}f = (3x^2,4y,-1)$. Thus, the tangent plane is $3(x - 1) + 4(y - 1) - (z - 3) = 0$ or $3x + 4y - z = 4$.

13. The relationship we use is $F_x(x,y)(dx/dt) + F_y(x,y)(dy/dt) = 0$. $F_x = 2x + y$ and $F_y = x + 2y$, so $(2x + y)(dx/dt) + (x + 2y)(dy/dt) = 0$.

17. Use the formula $dy/dx = -(\partial F/\partial x)/(\partial F/\partial y)$. $\partial F/\partial x = 1$ and $\partial F/\partial y = -\sin y$, so at $(1,\pi/2)$, $dy/dx = -1/(-1) = 1$.

21. The critical points occur where $\partial f/\partial x = \partial f/\partial y = 0$. Use the second derivative test to classify the critical points. $\partial f/\partial x = 2x - 6y$ and $\partial f/\partial y = -6x - 2y$. The only critical point is $(0,0)$. $\partial^2 f/\partial x^2 = 2$, $\partial^2 f/\partial y^2 = -2$, and $\partial^2 f/\partial x \partial y = -6$. Thus, $AC - B^2 = -32 < 0$, so $(0,0)$ is a saddle point.

25. $z_x = (12x^3 - 12x^2 - 24x)/12(1 + 4y^2) = x(x^2 - x - 2)/(1 + 4y^2) = x(x - 2)(x + 1)/(1 + 4y^2) = 0$ if $x = -1$, 0 , or 2 . Also, $z_y = -2y(3x^4 - 4x^3 - 12x^2 + 18)/3(1 + 4y^2)^2 = 0$ if $y = 0$. Therefore critical points occur at $(-1,0)$, $(0,0)$, and $(2,0)$. Further,

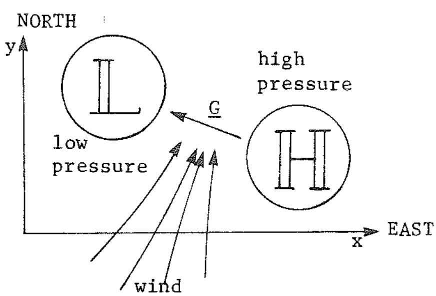
25. (continued)

$z_{xx} = (3x^2 - 2x - 2)/(1 + 4y^2)$; $z_{yy} = (-2/3)(3x^4 - 4x^3 - 12x^2 + 18) \times [(1 + 4y^2)^{-2} - 2(1 + 4y^2)^{-3}(8y^2)] = (-2/3)(3x^4 - 4x^3 - 12x^2 + 18)(1 + 4y^2 - 16y^2)/(1 + 4y^2)^3 = (-2/3)(3x^4 - 4x^3 - 12x^2 + 18)(1 - 12y^2)/ 1 + 4y^2)^3$; and $z_{xy} = -x(x - 2)(x + 1)(8y)/(1 + 4y^2)^2$. At $(-1,0)$, $z_{xx} = (3 + 2 - 2)/1 = 3$; $z_{yy} = (-2/3)(3 + 4 - 12 + 18) = (-2/3)(13) = -26/3$; and $z_{xy} = 0$. Hence $z_{xx}z_{yy} - z_{xy}^2 = -26 < 0$, so $(-1,0)$ is a saddle point. At $(0,0)$, $z_{xx} = -2$; $z_{yy} = -12$; and $z_{xy} = 0$, so $z_{xx}z_{yy} - z_{xy}^2 = 24 > 0$, so $(0,0)$ is a local maximum point. Finally, at $(2,0)$, $z_{xx} = 12 - 8 - 2 = 2$; $z_{yy} = (-2/3)(48 - 32 - 48 + 18) = (-2/3)(-14) = 28/3$; and $z_{xy} = 0$. Hence $z_{xx}z_{yy} - z_{xy}^2 = 56/3 > 0$, so $(2,0)$ is a local minimum point.

29. (a) $\|\underline{G}\| = ((\partial P/\partial x)^2 + (\partial P/\partial y)^2)^{1/2}$.

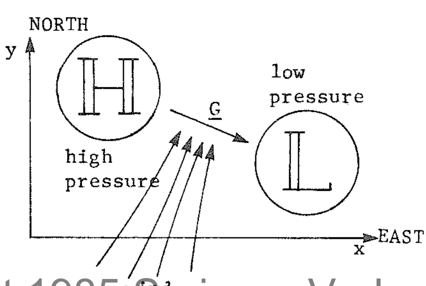
(b) $\underline{G}$ creates a force on the air mass, which produces a proportionate acceleration of the air mass in the same direction as $\underline{G}$, according to Newton's second law of motion.

(c)



This rotation of the earth turns the wind direction away from $\underline{G}$.

(d)



If, in the Southern Hemisphere, you stand with your back to the wind, the high pressure is on your <u>left</u> and the low pressure on your <u>right</u>.

33. It is easier to parametrize the circle. Then $f(x,y)$ becomes $h(t) = \cos^2 t - 2 \sin t \cos t + 2 \sin^2 t = 1 - 2 \sin t \cos t + \sin^2 t$. $h'(t) = -2 \cos^2 t + 2 \sin^2 t + 2 \sin t \cos t = -2 \cos 2t + \sin 2t$. This is 0 if $2 \cos 2t = \sin 2t$, i.e., $2 = \tan 2t$. Thus, $t = 0.554$, 2.124 , 3.695 , and 5.266 . $h(0.554) = h(2.124) \approx 0.382$ and $h(2.124) = h(5.266) \approx 2.618$. These are the two extreme values.

37. Use Lagrange multipliers. Let $f(x,y,\lambda) = x + 2y \sec \theta + \lambda(xy + y^2 \tan \theta - A)$. Then $f_x = 1 + \lambda y$; $f_y = 2 \sec \theta + \lambda(x + 2y \tan \theta)$; and $f_\lambda = xy + y^2 \tan \theta - A$. From $f_x = 0$, we get $\lambda = -1/y$; whereas from $f_y = 0$, we get $\lambda = -2 \sec \theta/(x + 2y \tan \theta)$. Hence $2y = (x + 2y \tan \theta)/\sec \theta$, so $2y(1 - \sin \theta) = x \cos \theta$. Thus, $x = 2y(\sec \theta - \tan \theta)$. Substitute this into $f_\lambda = 0$ to get $2y^2(\sec \theta - \tan \theta) + y^2 \tan \theta = A$. Then $y^2(2 \sec \theta - \tan \theta) = A$, so $y^2 = A/(2 \sec \theta - \tan \theta) = A \cos \theta/(2 - \sin \theta)$.

41. Let P_1 and P_2 denote the desired tangent planes.

(a) At $(1,1,2)$, $f_x = 2x = 2$; $f_y = 2y = 2$; $f_z = 2z = 4$. Hence P_1 is $2(x - 1) + 2(y - 1) + 4(z - 2) = 0$; i.e., $x + y + 2z = 6$. One normal vector is $(1,1,2)$. $g_x = 4x = 4$; $g_y = 6y = 6$; $g_z = 2z = 4$; so P_2 is $4(x - 1) + 6(y - 1) + 4(z - 2) = 0$. Rewrite it as $2x + 3y + 2z = 9$. Then one normal vector is $(2,3,2)$.

(b) Let θ denote the angle between P_1 and P_2 . Then $(1,1,2)\cdot(2,3,2) = \|(1,1,2)\|\|(2,3,2)\| \cos \theta$. Solve for $\theta = \cos^{-1}((2 + 3 + 4)/\sqrt{(1 + 1 + 4)(4 + 9 + 4)}) = \cos^{-1}(9/\sqrt{6\cdot 17}) = \cos^{-1}(3\sqrt{102}/34) \approx 0.47$.

41. (c) Let (a,b,c) denote the direction numbers of the desired line. This line must be perpendicular to both normal vectors, so $(a,b,c)\cdot(1,1,2) = 0$ and $(a,b,c)\cdot(2,3,2) = 0$. These give $a + b + 2c = 0$ and $2a + 3b + 2c = 0$, so $-2c = a + b = 2a + 3b$. Hence $b = -a/2$. Substitute for b in the first equation to get $a/2 + 2c = 0$, meaning $c = -a/4$. The scale of the direction numbers is arbitrary, so let $a = -4$. Then $b = 2$ and $c = 1$, and the line is $(x - 1)/(-4) = (y - 1)/2 = (z - 2)$. In vector form, it is $(1,1,2) + t(-4,2,1)$.

45. We want to minimize $C = \ell_1(a + bD_1) + \ell_2(a + bD_2)$, where ℓ_1, ℓ_2, D_1, and D_2 are the variables. Using the method of Lagrange multipliers, we let $f(\ell_1,\ell_2,D_1,D_2,\lambda_1,\lambda_2) = [\ell_1(a + bD_1) + \ell_2(a + bD_2) - C] - \lambda_1(\ell_1 + \ell_2 - \ell) - \lambda_2[kQ^m(\ell_1/D_1^{m_1} + \ell_2/D_2^{m_2}) - h]$. Taking partials, we get $\partial f/\partial\ell_1 = a + bD_1 - \lambda_1 - \lambda_2 kQ^m/D_1^{m_1}$; $\partial f/\partial\ell_2 = a + bD_2 - \lambda_1 - \lambda_2 kQ^m/D_2^{m_2}$; $\partial f/\partial D_1 = \ell_1 b + m_1\lambda_2 kQ^m\ell_1/D_1^{m_1+1}$; $\partial f/\partial D_2 = \ell_2 b + m_2\lambda_2 kQ^m\ell_2/D_2^{m_2+1}$; $\partial f/\partial\lambda_1 = \ell_1 + \ell_2 - \ell$; and $\partial f/\partial\lambda_2 = kQ^m(\ell_1/D_1^{m_1} + \ell_2/D_2^{m_2})$. To minimize C, we set all of the partials equal to zero. From $\partial f/\partial D_1 = 0$ and $\partial f/\partial D_2 = 0$, we get the simplified equations $b = -m_1\lambda_2 kQ^m/D_1^{m_1+1} = -m_2\lambda_2 kQ^m/D_2^{m_2+1}$. If $m_1 = m_2$, then $D_1 = D_2$.

If $m_1 \neq m_2$, we rearrange the last equation to get $1 = (m_1/m_2) \times (D_2^{m_2+1}/D_1^{m_1+1})$ or $m_1 D_2^{m_2+1} = m_2 D_1^{m_1+1}$. From $\partial f/\partial\ell_1 = 0$ and $\partial f/\partial\ell_2 = 0$, we simplify to $bD_2 = bD_1 - \lambda_2 kQ^m/D_1^{m_1} + \lambda_2 kQ^m/D_2^{m_2}$. Putting everything over the common denominator $D_1^{m_1}D_2^{m_2}$, we get $bD_1^{m_1}D_2^{m_2+1} = bD_2^{m_2}D_1^{m_1+1}$. Division by $bD_1^{m_1}D_2^{m_2}$ yields $D_2 = D_1$. Thus, $D_1 = D_2$ independent of m_1 and m_2.

49. We look for the maximum on the circle of radius 2 centered at the origin. The maximum is about 570 in the first quadrant.

53. (a) $dy/dx = -(\partial F/\partial x)/(\partial F/\partial y) = -(2x)/(3y^2 + e^y)$.

(b) By the chain rule, we have $(\partial F_1/\partial x)(dx/dx) + (\partial F_1/\partial y_1)(dy_1/dx) + (\partial F_1/\partial y_2)(dy_2/dx) = 0$ and $(\partial F_2/\partial x)(dx/dx) + (\partial F_2/\partial y_1)(dy_1/dx) + (\partial F_2/\partial y_2)(dy_2/dx) = 0$. Using the fact that $dx/dx = 1$, multiply the first equation by $\partial F_2/\partial y_2$ and the second by $-\partial F_1/\partial y_2$, and add. This yields $(\partial F_2/\partial y_2)(\partial F_1/\partial x) - (\partial F_1/\partial y_2)(\partial F_2/\partial x) + [(\partial F_2/\partial y_2)(\partial F_1/\partial y_1) - (\partial F_2/\partial y_1)(\partial F_1/\partial y_2)](dy_1/dx)$. Rearrangement gives $dy_1/dx = [(\partial F_1/\partial y_2)(\partial F_2/\partial x) - (\partial F_2/\partial y_2)(\partial F_1/\partial x)]/[(\partial F_2/\partial y_2)(\partial F_1/\partial y_1) - (\partial F_2/\partial y_1)(\partial F_1/\partial y_2)]$.

Similarly, multiply the first equation by $\partial F_2/\partial y_1$ and the second by $-\partial F_1/\partial y_1$. This yields $dy_2/dx = [(\partial F_1/\partial y_1)(\partial F_2/dx) - (\partial F_1/\partial x)(\partial F_2/\partial y_1)]/[(\partial F_1/\partial y_2)(\partial F_2/\partial y_1) - (\partial F_2/\partial y_2)(\partial F_1/\partial y_1)]$.

(c) Let $F_1 = x^2 + y_1^2 - \cos x$ and $F_2 = x^2 - y_2^2 - \sin x$. Then $\partial F_1/\partial x = 2x + \sin x$, $\partial F_1/\partial y_1 = 2y_1$, $\partial F_1/\partial y_2 = 0$, $\partial F_2/\partial x = 2x - \cos x$, $\partial F_2/\partial y_1 = 0$, and $\partial F_2/\partial y_2 = 2y_2$. Thus, $dy_1/dx = [0 - 2y_2(2x + \sin x)]/[4y_1y_2 - 0] = -(2x + \sin x)/2y_1$ and $dy_2/dx = [2y_1(2x - \cos x) - 0]/[0 - 4y_1y_2] = (\cos x - 2x)/y_2$.

57. (a) $P_y = 2x^2y\exp(xy^2) + (x^2y^2 + 2x)2xy\exp(xy^2) = (2x^3y^3 + 6x^2y) \times \exp(xy^2)$ and $Q_x = 6x^2y\exp(xy^2) + (2x^3y)y^2\exp(xy^2) = P_y$; therefore, a function exists. Integrate $f_x = (x^2y^2 + 2x)\exp(xy^2)$ by parts with $u = x^2y^2 + 2x$ and $v = (1/y^2)\exp(xy^2)$ to get $(x^2 + 2x/y^2)\exp(xy^2) - \int(2x + 2/y^2)\exp(xy^2)dx$. Use parts again to get $(x^2 + 2x/y^2)\exp(xy^2) - (2x/y^2 + 2/y^4)\exp(xy^2) + \int(2/y^2)\exp(xy^2)\,dx = x^2\exp(xy^2) + g(y)$ for some $g(y)$. Integrate f_y by substituting $u = xy^2$ to get $\int 2x^3y\exp(xy^2)dy = \int x^2e^u du = x^2e^u + h(x) = x^2\exp(xy^2) + h(x)$ for some $h(x)$. Clearly, $g(y) = h(x) = $ constant, so $f(x,y) = x^2\exp(xy^2) + C$ is the solution.

57. (b) $P_y = (2xy + 3xy^2)e^{xy^3}$ and $Q_x = 2y(3x^2 + y^3)e^{xy^3}$. Since $P_y \neq Q_x$, no such function exists.

(c) Integrate $f_x = 2x/(1 + x^2 + y^2)$ to get $f(x,y) = \ln(1 + x^2 + y^2) + g(y)$, for some $g(y)$. Integrate $f_y = 2y/(1 + x^2 + y^2) = \ln(1 + x^2 + y^2) + h(x)$, for some $h(x)$. Clearly, $h(x) = g(y) =$ constant , so $f(x,y) = \ln(1 + x^2 + y^2) + C$ is the function.

(d) Let $A = (1 + x^2 + y^2)$. Then $P_y = [2A - 4y^2]/A^2$ and $Q_x = [2A - 4x^2]/A^2$. Since $P_y \neq Q_x$, there is no such function.

61. (a) $f_x = -2xy^3/(x^2 + y^2)^2$ if $(x,y) \neq (0,0)$. $f_y = [3y^2(x^2 + y^2) - 2y^4]/(x^2 + y^2)^2 = y^2(3x^2 + y^2)/(x^2 + y^2)^2$ if $(x,y) \neq (0,0)$. In order to deal with $f_x(0,0)$ and $f_y(0,0)$, approach the origin along the line $y = 0$ for f_x and along the line $x = 0$ for f_y . Hence, $f_x(0,0) = \lim_{x\to0}[-2x\cdot0/x^4] = \lim_{x\to0}(0/x^4) = 0$; and $f_y(0,0) = \lim_{y\to0}[y^2(0 + y^2)/y^4] = \lim_{y\to0}(y^4/y^4) = 1$.

(b) We have $x = r\cos\theta$ and $y = r\sin\theta$, so $\partial x/\partial r = \cos\theta$ and $\partial y/\partial r = \sin\theta$. Compute $(\partial/\partial r)f(r\cos\theta, y\sin\theta) = (\partial/\partial r)f(x,y) = f_x(\partial x/\partial r) + f_y(\partial y/\partial r) = -2xy^3\cos\theta/(x^2 + y^2)^2 + y^2(3x^2 + y^2)/(x^2 + y^2)^2 = -2r^4\cos^2\theta\sin^3\theta + 3r^4\sin^2\theta\cos^2\theta + r^4\cos^2\theta/r^4 = \cos^2\theta(-2\sin^3\theta + 3\sin^2\theta + 1)$. This quantity is defined for all θ (and is 1 when $\theta = 0$) .

(c) Let D_θ denote the directional derivative in the direction θ at the origin. Suppose $D_\theta = \underline{\nabla} f\cdot(\cos\theta, \sin\theta) = f_x\cos\theta + f_y\sin\theta = \sin\theta$. However, this disagrees with (b). This result does not contradict the chain rule because the partials are not continuous at $(0,0)$.

TEST FOR CHAPTER 16

1. True or false.

(a) If a two-variable function f is defined throughout the disk $x^2 + y^2 \leqslant 1$, then f must have a minimum and a maximum, even if a saddle point occurs at the origin.

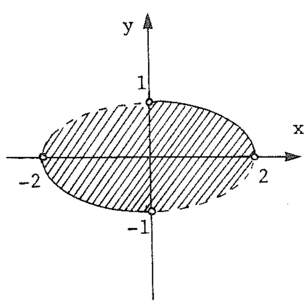
(b) If F is a function of x and y such that $y = f(x)$, then $dy/dx = (\partial F/\partial x)/(\partial F/\partial y)$.

(c) The gradient is a directional derivative.

(d) If f is a differentiable function and a local minimum occurs at (x_0,y_0), then $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$.

(e) If the gradient $\underline{\nabla}f(x_0,y_0)$ is nonzero, it is normal to the surface $z = f(x,y)$ at (x_0,y_0,z_0).

2.



Locate the minimum and maximum (if they exist) for $f(x,y) = xy^2 + x^2 - 1$ over the region shown at the left. The boundary is the ellipse $x^2/4 + y^2 = 1$, but it is part of the region only in the first and third quadrants.

3. (a) Find the maximum of xyz subject to the constraints $x + y + z = 32$ and $x - y + z = 0$.

(b) Explain what this means geometrically.

4. Classify the critical points of $f(x,y) = x^3 - 3xy + y^3 - 5$.

5. (a) What is the relationship between the gradient and the tangent plane?

(b) Compute the tangent plane to the surface $3x^3 - 9xy - 3z^2y + z = 26$ at the point $(1,-1,2)$.

6. Let $f(x,y,z) = e^{xy} + e^{xz} + e^{2yz}$.

   (a) Compute the directional derivative of f at $(1,1,1)$ in the direction $\underline{i} + 2\underline{j} - 2\underline{k}$.

   (b) Explain what the answer in part (a) tells you.

   (c) In what direction is f increasing most rapidly at $(1,1,1)$?

7. A rectangular box has sides with length x , y , and $2x$. Given 100 square units of surface area, what is the maximum volume that can be held by the box? Use Lagrange multipliers.

8. Let $z = F(x,y) = \cos(x^2y^2) + 2xy$.

   (a) How is dy/dx related to F_x and F_y ?

   (b) Use the formula in (a) to compute dy/dx .

   (c) Compute the directional derivative of F at $(1,\sqrt{\pi})$ in the direction of $\underline{i} - \underline{j}$.

9. Find the points on the plane $5x + 2y - z = 3$ which are closest and farthest from the point $(3,0,2)$.

10. Your worst enemy has cursed you with $f(x,y)$ years of bad luck after you "accidentally" stepped on his foot. However, being a sporting human being, he offers you a chance to determine your own curse. He owns a device which randomly selects a value for $f(x,y) = x^2 + \sin y$ over the rectangle $1 \leqslant x \leqslant 3$ and $\pi/2 \leqslant y \leqslant \pi$. Your worst enemy offers you a chance to push the button to determine your curse. Find the minimum and maximum number of years your worst enemy plans to curse you with bad luck.

ANSWERS TO CHAPTER TEST

1. (a) True

   (b) False; negative sign is missing.

   (c) False; the gradient is a vector; the directional derivative is a scalar.

   (d) True

   (e) True

2. Minimum at $(0,0)$; no maximum.

3. (a) 1024 at $(8,16,8)$.

   (b) It is the maximum of xyz on the line which forms the intersection of the planes $x + y + z = 32$ and $x - y + z = 0$.

4. Saddle point at $(0,0)$; local minimum at $(1,1)$.

5. (a) The tangent plane is $\underline{\nabla} f(\underline{r}_0)\cdot(\underline{r} - \underline{r}_0) = 0$ where $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$.

   (b) $18x - 21y + 13z = 65$

6. (a) $2e/3$

   (b) If one moves one unit in direction $\underline{i} + 2\underline{j} - 2\underline{k}$, f should increase $2e/3$ units.

   (c) $2e\underline{i} + (e + 2e^2)\underline{j} + (e + 2e^2)\underline{k}$

7. $x = 5/\sqrt{3}$; $y = 20/3\sqrt{3}$.

8. (a) $dy/dx = -F_x/F_y$

   (b) $[1 - xy \sin(x^2y^2)]/[xy \sin(x^2y^2) - 1]$

   (c) $\sqrt{2\pi} - \sqrt{2}$

9. Closest: $(4/3,-2/3,43/15)$. Farthest: none.

10. Minimum $= 1$; maximum $= 10$.